Beauty is a Beast, Frog is a Prince:
Assortative Matching with Nontransferabilities*

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Abstract

We present sufficient conditions for monotone matching in environments where utility is not fully transferable between partners. These conditions involve complementarity in types not only of the total payoff to a match, as in the transferable utility case, but also in the degree of transferability between partners. We apply our conditions to study some models of risk sharing and incentive problems.

Keywords: Nontransferable utility, assignment games, sorting

1 Introduction

For the economist analyzing household behavior, firm formation, or the labor market, the characteristics of matched partners are paramount. The educational background of men and women who are married, the financial positions of firms that are merging, or the productivities of agents who are working together, all matter for understanding their respective markets. Matching patterns serve as direct evidence for theory, figure in the econometrics of selection effects, facilitate theoretical analysis, and are even treated as policy variables.

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Much is known about characterizing matching in the special case of transferable utility (TU). For instance, if the function representing the total payoff to the match satisfies increasing (decreasing) differences in the partners' attributes, then there will always be positive (negative) assortative matching, whatever the distribution of types. Because they are distribution-free, results of this sort are very powerful and easy to apply.

For an individual contemplating marriage, a firm entering into a joint venture, or a film producer seeking a director, the partners' characteristics are also crucial, for two equally important and possibly conflicting reasons: they determine the gains from the relationship, and they affect the ability to share in them. A star director might make for a profitable collaboration, but if stars are cagey or obstinate, it will be too costly or unpleasant to keep the film under budget, and the producer might go for someone less well-known or talented.

Concerns about this effect – imperfect transferability – are not limited to people in the real world. In much of economic analysis, the utility among individuals is not fully transferable ("non-transferable," or NTU, in the parlance).[1] Partners may be risk averse with limited insurance possibilities; incentive or enforcement problems may restrict the way in which the joint output can be divided; or policy makers may impose rules about how output may be shared within relationships. As Becker (1973) pointed out long ago, rigidities that prevent partners from costlessly dividing the gains from a match may change the matching outcome, even if the level of output continues to satisfy monotone differences in type.

While interest in the issues represented by the non-transferable case is both long-standing and lively (see for instance Farrell-Scotchmer, 1988 on production in partnerships; Rosenzweig-Stark, 1989 on risk sharing in households; and more recently, Lazear, 2000 on incentive schemes for workers; Ackerberg-Botticini, 2002 on sharecropping; and Chiappori-Salanié, 2003 on the empirics of contracts), for the analyst seeking to characterize the equilibrium matching pattern in such settings, there is little theoretical guidance.

The purpose of this paper is to offer some. We present sufficient conditions for assortative matching that are simple to express, intuitive to understand, and, we hope, tractable to apply. We illustrate their use with some examples that are of independent interest.

The class of models we consider are two-person matching games without

[1] This terminology dates from the 1950's and refers to all models that depart from the transferable utility assumption. In some circles, the term nontransferable has been used to refer solely to the extreme situation in which there is *no* possibility for making transfers; this special case will not occupy much of our attention here. Smith (2002) offers an analysis of that case, with particular attention devoted to search frictions.

search frictions in which the utility possibility frontier for any pair of agents is a strictly decreasing function. After introducing the model and providing formal definitions of the monotone matching patterns, we review the logic of the classical transferable utility results, a close of examination of which leads us to propose the "generalized difference conditions" (GDC) that suffice to guarantee monotone matching for any type distribution (Proposition 1). We then apply them to a simple model of risk sharing within households.

As it is often easier to verify properties of functions locally than globally, we also present differential conditions for monotone matching (Proposition 2). Though stronger than the GDC, the differential conditions offer additional insight into the forces governing matching. In particular, they highlight the role not only of the complementarity in partners' types that figures in the TU case, but also of *complementarity between type and the degree of transferability* (slope of the frontier) that is the new feature in the NTU case. Even if the output satisfies increasing differences in types, failure of the type-transferability complementarity – as happens if higher types are more "difficult" than lower ones – may overturn the predictions of the TU model and lead instead to negative assortative matching or some more complex and/or distribution-dependent pattern.

We use the differential conditions to study a model in which principals with different monitoring technologies are matched to agents with different wealths, one interpretation of which may address some puzzling results concerning the assignment of peasants to crop types in the empirical sharecropping literature. In the example, the type-transferability relationship is responsible for the predicted matching pattern, which goes in a (possibly) unexpected direction.

We then go on to discuss other techniques that facilitate application of the generalized difference conditions. For instance, the truth of the GDC depends only on the ordinal properties of preferences (Proposition 4); this fact broadens the scope of applicability of the local conditions (Corollary 1). We also consider the relationship between the GDC and more familiar difference conditions, including lattice theoretic notions (Propositions 5 and 6), and devote some attention to necessary conditions for monotone matching (Proposition 7).

The next section delves further into the ideas underlying the general theoretical analysis by examining a very simple example. It then introduces the two models that will be used to illustrate the application of our results.

2 Issues and Examples

How do nontransferabilities affect the matching pattern? Consider the following example, which is inspired by the one in Becker (1973).

Example 1 *Suppose there are two types of men, $l < h$, and two types of women, $L < H$. The total "output" they produce when matched, as a function of the partners' types, is described by the matrix*

$$\begin{array}{ccc} & L & H \\ l & 4 & 7 \\ h & 7 & 9 \end{array}.$$

Note that the output function satisfies decreasing differences (DD)*, since $9 - 7 < 7 - 4$. If utility is fully transferable, then it is well known that decreasing differences implies that a stable outcome will always involve negative assortative matching (NAM): high types will match with low types. If to the contrary we had a positive match of the form $\langle l, L\rangle$, $\langle h, H\rangle$ with equilibrium payoffs (u_l, u_L) and (u_h, u_H), then there would always be a pair of types that could do strictly better for themselves: $u_l + u_h = (4 - u_L) + (9 - u_H) < (7 - u_H) + (7 - u_L)$; thus $u_l < 7 - u_H$ or $u_h < 7 - u_L$; l could offer H (or h could offer L) slightly more than her current payoff and still get more for himself, destabilizing the positive match. The negative matching outcome maximizes total output.*

Suppose instead that utility is not perfectly transferable, and consider the extreme case in which any departure from equal sharing within the marriage is impossible. For instance, the payoff to the marriage could be generated by the joint consumption of a local public good. Thus each partner in $\langle h, H\rangle$ gets 4.5, each in $\langle h, L\rangle$ gets 3.5, etc. Now the only stable outcome is positive assortative matching (PAM): each h is better off matching with H (4.5) than with L (3.5), and thus the "power couple" blocks a negative assortative match. As Becker noted, with nontransferability, the match changes, and aggregate performance suffers as well.

Of course this extreme form of nontransferability is not representative of most situations of economic interest, and we wonder what happens in the intermediate cases.

Example 2 *Modify the previous example by introducing a dose of transferability: some compensation, say through the return of favors, makes it possible to depart from equal sharing. Consider two simple cases. In the first, the high types are "difficult," while the low types are "easy,": beauty is a beast,*

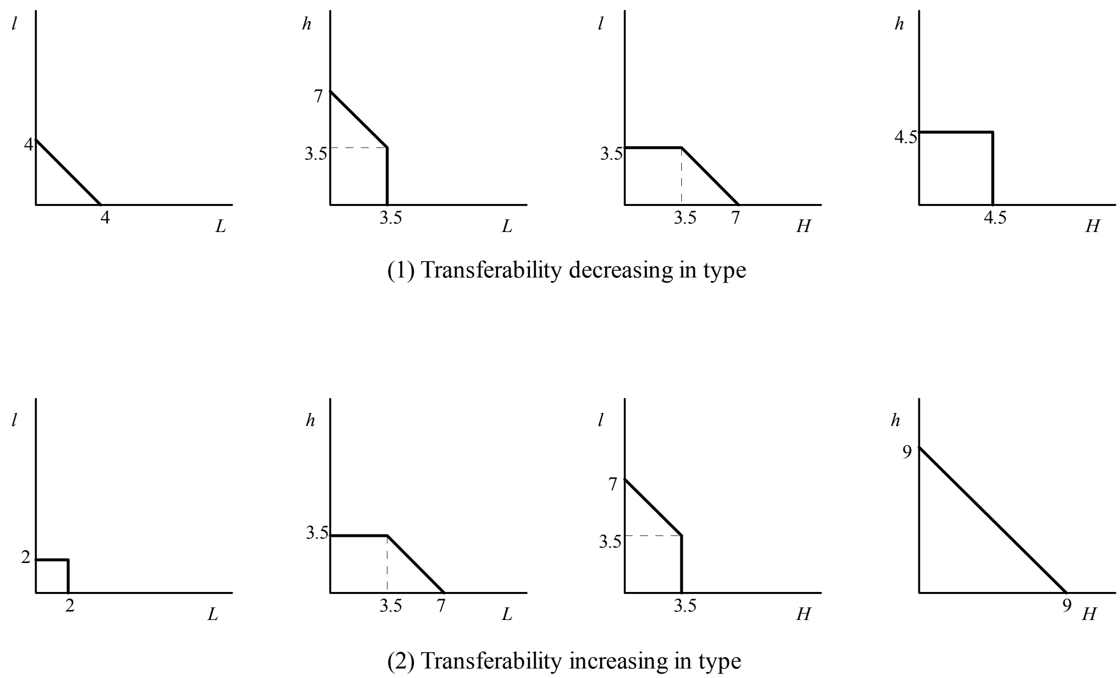

Figure 1: Utility possibility frontiers for Example 2.

frog is a prince. That is, utility is perfectly transferable between l and L, l can transfer to H, but not vice versa, and L can transfer to h but not vice versa. In the second case, the high types are easy and the low types difficult. See Figure 1, which depicts the utility possibility frontiers between pairs of types, assuming feasible transfers are made starting from the equal sharing point.

In the first case, the degree of transferability is decreasing *in type, and in particular is changing in the same direction as (marginal) productivity. The unique outcome is NAM in this case: if things were otherwise, a high type could promise a low type almost 2.5, garnering a bit over 4.5 for itself, and the low type will be happy to accept the offer (the only way this could not happen is if both l and L were getting at least 2.5, which is an impossibility).*

In the second case, the degree of transferability is increasing *in type, opposite the direction that productivity increases, and this opposition between productivity and transferability is enough to overturn the TU outcome. The easygoing high types now can get no more than 3.5 out of a mixed relationship, so they prefer a match with each other, wherein 4.5 would be available to each.*

The basic intuitions contained in this second example carry over to the general case, and are in a nutshell the content of our main results, Propositions 1 and 2.[2] Transferability, and its dependence on type, can be as important as productivity in determining the nature of sorting.

In the remainder of this section we present two less-contrived examples that are representative of those considered in the literature. The first is a marriage market model in which partners vary in initial wealth (and risk attitude) and must share risks within their households. Although this topic has attracted considerable attention in the development literature and economics of the family, we are not aware of any attempts to establish formally what the pattern of matching among agents with differing risk attitudes would be, something which is obviously important for empirical identification, say of risk-sharing versus income-generation motives for marriage and migration (Rosenzweig-Stark, 1989).

The second is a principal-agent model in which agents vary in their initial wealth (and therefore risk aversion), and principals vary in their ability to monitor agents. Sorting effects in this sort of model are of direct interest in some applications (e.g., Newman, 1999; Prendergast, 2002) and are important considerations in the econometrics of contracting (Ackerberg-Botticini, 2002).

Example 3 *(Risk sharing in households). Consider a stylized marriage market model in which the primary desideratum in choosing a mate is suitability for risk sharing. We ignore gender in what follows, i.e., study a "one-sided" model.*

Suppose that household production is random, with a finite number of possible outcomes $w_i > 0$ *and associated probabilities* π_i*. Each individual initially has one unit of wealth; upon marriage, each receives a monetary wedding gift from its parents, which is assumed to be proportional to the parents' wealth. Everyone is an expected utility maximizer; income* y *yields utility* $\ln y$*, and an individual's type is the wealth* $a > 0$ *received as the wedding gift. Unmatched agents receive no gift and therefore get utility zero. For informational or enforcement reasons, the only risk sharing possibilities in this economy lie within a household consisting of two agents. When partners match, their (explicit or implicit) contract specifies how each realization of the output will be shared between them.*

[2]The only difference is that we shall require the frontiers to be strictly decreasing; the above examples could easily be modified to conform to this requirement without changing any conclusion.

Write the utility possibility frontier for a match between individuals of types a and b as a function ϕ of b's payoff v; it is generated by solving the optimal risk sharing problem:

$$\phi(a,b,v) \equiv \max_{\{x_i\}} \Sigma_i \pi_i \ln(1+a+w_i-x_i) \; s.t. \; \Sigma_i \pi_i \ln(1+b+x_i) \geq v, \quad (1)$$

Note that the wealth, including that received from the parents, can be transferred as part of the share in this set-up. The first-order condition (known as Borch's rule) is $\frac{1}{1+a+w_i-x_i} = \lambda \frac{1}{1+b+x_i}$*, where* λ *is the multiplier on the constraint, from which one solves for the optimal sharing rule:*

$$x_i = (w_i+a+b+2)e^{v-\Sigma_i \pi_i \ln(w_i+a+b+2)} - b - 1.$$

This yields

$$\phi(a,b,v) = \ln(1-e^{v-\Sigma_i \pi_i \ln(w_i+a+b+2)}) + \Sigma_i \pi_i \ln(w_i+a+b+2). \quad (2)$$

Clearly, this function is not linear in v, so utility is only imperfectly transferable: the cost to a of transferring a small amount to b depends on how much each partner already has. The same is true of the following example.

Example 4 *(Matching principals and agents). Principals, who differ in their ability to monitor effort, must match with agents, who differ in initial wealth and therefore risk aversion. The question is whether the most closely monitored tasks, which can be compensated via low-risk contracts, are accepted by the most or least risk averse, i.e. the poorest or wealthiest agents. Possible interpretations in include the occupational distinction between entrepreneurs and workers (the former bear much risk, the latter little or none), the assignment of fund managers to different portfolios, or the assignment of crop varieties to peasants with different wealth levels.*

There is a continuum of risk-neutral principals with type indexed by $p \in [\underline{p}, 1]$, $\underline{p} \in (\frac{1}{2}, 1)$, *and an equal measure of agents with type index* $a > 1$. *The principal's type is the probability that his agent's effort e, which can either be* 1 *or* 0, *is correctly detected on his task. All tasks are equally productive, yielding expected revenue* π *when effort is high, and every principal wishes to implement* $e = 1$ *(this amounts to assuming that* $\underline{p}$ *is sufficiently high). All agents derive utility* $\ln y$ *from income y; their type represents initial wealth.*

The frontier for a principal of type p who is matched to an agent of type a is given by

$$\phi(p,a,v) = \max \; \pi - pw_1 - (1-p)w_0$$
$$s.t. \; p\ln(a+w_1) + (1-p)\ln(a+w_0) - 1 \geq v$$
$$p\ln(a+w_1) + (1-p)\ln(a+w_0) - 1 \geq (1-p)\ln(a+w_1) + p\ln(a+w_0),$$

where w_1 and w_0 are the wages paid in case the signal of effort is 1 or 0 respectively. The second inequality is the incentive compatibility condition that ensures the agent takes high effort. The frontier for an agent of type a matched to a principal of type p who gets v is

$$\phi(a, p, v) = \max \; p \ln(a + w_1) + (1 - p) \ln(a + w_0) - 1$$
$$\pi - pw_1 - (1 - p)w_0 \; s.t. \geq v$$
$$p \ln(a + w_1) + (1 - p) \ln(a + w_0) - 1 \geq (1 - p) \ln(a + w_1) + p \ln(a + w_0),$$

The solution to these problems yields

$$\phi(p, a, v) = \pi + a - e^{v+1}[pe^{\frac{1-p}{2p-1}} + (1 - p)e^{-\frac{p}{2p-1}}] \tag{3}$$

and

$$\phi(a, p, v) = \frac{1 - p}{2p - 1} + \ln\left(\frac{\pi + a - v}{pe^{\frac{1}{2p-1}} + 1 - p}\right) \tag{4}$$

Intuition might suggest that since wealthier agents are less risk averse, they should be matched to tasks for which the signal quality is poor, since these tasks are effectively riskier. Indeed, when $p = 1$, the optimal contract is a fixed wage, since in equilibrium the agent will always generate the high effort signal, while for lower values of p the agent must bear some income risk. As we shall see in Section 5.1 , this intuition is incomplete, and indeed misleading, and the complete analysis can offer an explanation for some seemingly puzzling results in the empirical literature.

As with expression (2), (3) and (4) depict nontransferable utility models in which the frontiers, though downward sloping, do not have constant unit slope. As we have shown, the traditional techniques for determining matching patterns do not apply in these cases. We shall revisit these examples as we present our general results in order to illustrate their application. As it turns out, these examples are both solvable by a variety of techniques.

3 Theoretical Preliminaries

The economy is populated by a continuum of agents who differ in type, which is taken to be a real-valued attribute such as skill, wealth, or risk attitude. In the *two-sided* model, agents are also distinguished by a binary "gender" (man-woman, firm-worker, etc.). Payoffs exceeding that obtained in autarchy, which for the general analysis we normalize to zero for all types,[3] are generated only if agents of opposite gender match. In the *one-sided* model, there is no gender distinction, but positive payoffs still require a match (in neither case is there any additional gain to matching with more than one other agent). For simplicity, we will assume that the measure of agents on each side of a two-sided model is equal. The type space A is a compact subset of the real line (or such a set crossed with $\{0, 1\}$ in the two-sided case[4]). The number of types may be finite or infinite, and we think of there being a continuum of each type.

The object of analytical interest to us is the utility possibility frontier (since in equilibrium agents will always select an allocation on this frontier) for each possible pairing of agents. This frontier will be represented by a *function* $\phi(a, b, v)$ which denotes the maximum utility generated by a type a in a match with a type b who receives utility v. We take ϕ to be a primitive of the model for the general analysis, although as in the examples we have presented, it will often be derived from more fundamental assumptions about technology, preferences and choices made by the partners after they match. We shall sometimes refer to the first argument of ϕ as "*own type*" and the third argument as "*payoff.*"

We assume throughout that this function is continuous and strictly decreasing in v and continuous in the types. If $\phi(a, b, v)$ can be written $f(a, b)-v$, we have transferable utility (TU); otherwise, we have nontransferable utility (NTU).

The maximum equilibrium payoff that a could ever get in a match with b is $\phi(a, b, 0)$, since b would never accept a negative payoff. By slight abuse of notation, if $v > \phi(b, a, 0)$, we will define $\phi(a, b, v) = 0$. Note that $\phi(a, b, v)$ is still strictly decreasing in $[0, \phi(b, a, 0)]$ and that $\phi(b, a, \phi(a, b, v)) = v$ for all v in this interval: $\phi(b, a, \cdot)$ and $\phi(a, b, \cdot)$ are inverses there. In general, of course, $\phi(a, b, v) \neq \phi(b, a, v)$.

The notation reflects two further assumptions of matching models, namely (1) that the payoff possibilities depend only on the types of the agents and not on their individual identities; and (2) the utility possibilities of the pair of agents do not depend on what other agents in the economy are doing, i.e., there are no externalities across coalitions.[5]

[3]In many applications, the autarchy payoff varies with type. For instance, in the principal-agent example it is natural to assume that an unmatched agent a gets $\ln a$. The analysis extends to this case almost without modification: see Section 5.5.

[4]In this case, it is to be understood that in comparing types, one only considers attributes of agents from the same side, i.e., "$a > b$ and $c > d$" entails that a and b are on one side, and c and d are on the other.

[5]Of course, in general, the equilibrium *payoffs* in one coalition will depend on the other

3.1 Equilibrium

It is useful to identify equilibrium with core allocations: we are interested in specifying the way types are matched and the payoff to each type. Specifically, an equilibrium consists of a matching correspondence $\mathfrak{M}^* : A \rightrightarrows A$ that specifies the type (s) to which each type is matched, and a payoff allocation $u^* : A \to \mathbb{R}$ specifying the equilibrium utility achieved by each type (Lemma 1 below ensures that u^* is a function; we will write u_a^* for its value at a). The key property it satisfies is a stability or no-blocking condition: if u^* is the equilibrium payoff allocation, then there is no pair a, b and payoff v such that $\phi(a, b, v) > u_a^*$ and $v > u_b^*$. In addition to this requirement, u^* must satisfy feasibility (if a and b are matched, then $u_a^* \leq \phi(a, b, u_b^*)$ and individual rationality $u_a^* \geq 0$ for all a), and $\mathfrak{M}^*$ must be *measure consistent*, i.e., the measure of first partners must equal the measure of second partners (this requirement arises because of the continuum of agents: it avoids situations in which, say, one one-millionth of the population matches one-to-one with the rest). Equilibria always exist under our assumptions.[6]

3.2 Descriptions of Equilibrium Matching Patterns

A match is a measurable correspondence

$$\mathfrak{M}^* : A \rightrightarrows A.$$

$\mathfrak{M}^*$ is symmetric: $a \in \mathfrak{M}^*(b)$ implies $b \in \mathfrak{M}^*(a)$. Let

$$\overline{A} = \{a \in A : \exists b \in \mathfrak{M}^*(a) : a \geq b\}$$

be the set of larger partners. Obviously, $\overline{A}$ depends on $\mathfrak{M}^*$, but we suppress this dependence in the notation. Note that in the case of two-sided matching, we identify $\overline{A}$ with one of the sides.

Symmetry of $\mathfrak{M}^*$ implies that the correspondence $\mathfrak{M}$

$$\mathfrak{M} : \overline{A} \rightrightarrows A, \text{where } b \in \mathfrak{M}(a) \Longleftrightarrow b \in \mathfrak{M}^*(a) \;\&\; a \geq b,$$

completely characterizes the match. The coalitions generated by $\mathfrak{M}^*$ can then be written as ordered pairs $\langle a, b \rangle \in \overline{A} \times \mathfrak{M}(\overline{A})$. Our descriptions of

coalitions.

[6]The facts that there is a continuum of agents and that the only coalitions that matter are singletons and pairs make the core here a special case of the f-core. See Kaneko-Wooders (1996) for definitions and existence results — with a continuum of types, they also assume that the slopes of the frontiers are uniformly bounded away from zero, a condition that is satisfied if the marginal utility of consumption at autarchy is not infinite.

matching patterns will be in terms of the properties of the graph of $\mathfrak{M}$. Note that for a one-sided model, the graph of $\mathfrak{M}$ is the portion of the graph of $\mathfrak{M}^*$ that is on or below the 45^0 line.

When $\mathfrak{M}$ is a monotone correspondence, matching is *monotone.* We consider only a few types of monotone matching patterns in this paper. An equilibrium satisfies *segregation* if $\mathfrak{M}(a) = \{a\}$ for all a. It satisfies *positive assortative matching* (PAM) if for all $a, b \in \overline{A}$, $a > b, c \in \mathfrak{M}(a), d \in \mathfrak{M}(b) \implies c \geq d$, and *negative assortative matching* (NAM) if for all $a, b \in \overline{A}$, $a > b, c \in \mathfrak{M}(a), d \in \mathfrak{M}(b) \implies c \leq d$. In one sided models, an alternative way to say that there is NAM is that whenever we have types $a > b \geq c > d$, $\langle a, c\rangle, \langle b, d\rangle$ and $\langle a, b\rangle, \langle c, d\rangle$, as well as segregation, are ruled out as possible matches (only $\langle a, d\rangle, \langle b, c\rangle$ is permitted).

Note that while segregation only occurs in one-sided models, PAM and NAM can occur in both one- and two-sided models. However, in this paper, when we refer to PAM, we shall be referring exclusively to two-sided models.

Say that an equilibrium is *payoff equivalent* to PAM if any four types that are not matched in a positive assortative way can be rearranged among themselves in a positive assortative way without changing their payoffs (from which it follows that the new match constructed this way, along with the original payoffs, is also an equilibrium). [7] Payoff equivalence to segregation and to NAM are defined analogously. For brevity, we will say that there is segregation, PAM, or NAM if the equilibrium satisfies the corresponding notion of payoff equivalence.

For our purposes, the important consequence of payoff equivalence is that if $a, b \in \overline{A}$ with $a > b$, $c > d$, $\langle a, d\rangle$ and$\langle b, c\rangle$ are matches in an equilibrium that is payoff equivalent to PAM, then $u_a^* = \phi(a, c, u_c^*)$ and $u_b^* = \phi(b, d, u_d^*)$.

[7]Formally, $(\mathfrak{M}, u^*)$ is payoff equivalent to PAM if whenever $a, b \in \overline{A}, a > b$, $c > d$, $d \in \mathfrak{M}(a)$ and $c \in \mathfrak{M}(b)$, there is another measure consistent match $\mathfrak{M}'$ with $\mathfrak{M}' = \mathfrak{M}$ on $\overline{A} \backslash \{a, b\}, d \in \mathfrak{M}'(b)$, $c \in \mathfrak{M}'(a)$ and either $d \notin \mathfrak{M}'(a)$ or $c \notin \mathfrak{M}'(b)$ such that u^* is feasible.

This falls short of saying that $\mathfrak{M}'$ satisfies PAM, but if the type distribution has finite support and rational values, a straightforward algorithmic argument shows that if $(\mathfrak{M}, u^*)$ is an equilbrium in which $\mathfrak{M}$ violates PAM but is payoff equivalent to it, then there is another measure-consistent match $\widehat{\mathfrak{M}}$ such that $(\widehat{\mathfrak{M}}, u^*)$ is an equilibrium that satisfies PAM. Under standard regularity conditions, a limit argument establishes the same thing for arbitrary distributions.

4 Sufficient Conditions for Monotone Matching

4.1 Logic of the TU Case

Before proceeding, let's recall the nature of the conventional transferable utility result and why it is true, as that will provide us with guidance to the general case. In the TU case, only the total payoff $f(a, b)$ is relevant. The assumption that is often made about f is that it satisfies *increasing differences* (ID): whenever $a > b$ and $c > d$, $f(c, a) - f(d, a) \geq f(c, b) - f(d, b)$. Why does this imply positive assortative matching (segregation in the one-sided case), irrespective of the distribution of types? Usually, the argument is made by noticing that the total output among the four types is maximized (a necessary condition of equilibrium in the TU case, but not, we should emphasize, in the case of NTU) when a matches with c and b with d: this is evident from rearranging the ID condition.

However, it is more instructive to analyze this from the equilibrium point of view. Suppose that a and b compete for the right to match with c rather than d. The increasing difference condition says that a can outbid b in this competition, since the incremental output produced if a switches to c exceeds that when b switches. In particular, this is true *whatever* the level of utility v that d might be receiving: rewrite ID as $f(c, a) - [f(d, a) - v] \geq f(c, b) - [f(d, b) - v]$; this is literally the statement that a's willingness to pay for c, given that d is getting v, exceeds b's. Thus a situation in which a is matched with d and b with c is never stable: a will be happy to offer more to c than the latter is getting with b.[8] The ID result is *distribution free*: the type distribution will affect the equilibrium payoffs, but the argument just given shows that a's partner must be larger than b's regardless of what those payoffs might be.

The convenient feature of TU is that if a outbids b at one level of v, he does so for all v. Such is not the case with NTU. Our sufficient condition will require explicitly that a can outbid b for *all* levels of v. If this requirement seems strong, recall that the nature of the result sought, namely monotone matching regardless of the distribution, is also strong. By the same token, it is weaker than ID, and includes TU as a special case.

In an NTU model, the division of the surplus between the partners cannot be separated from the level that they generate. Switching to a higher type

[8] This assumes that b prefers to be with c than with d in the first place – else b can upset the match himself – so if b is getting v' with c, $f(c, b) - v' < f(c, b) - [f(d, b) - v]$ follows from $v' > f(d, b) - v$

partner may not be attractive if it is also more costly to transfer utility to a high type, that is, if the frontier is steeper. A sufficient condition for PAM is that not only is there the usual complementarity in the production of surplus, but also there is a complementarity in the transfer of surplus – frontiers are flatter, as well as higher, for higher types. This will perhaps be more apparent from the local form of our conditions.

4.2 Generalized Difference Conditions

Let $a > b$ and $c > d$ and suppose that d were to get v. Then the above reasoning would suggest that a would be able to outbid b for c if

$$\phi(c, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v)). \tag{5}$$

The left-hand side is a's willingness to "pay" (in utility terms) for c rather than d, given that d receives v (a then receives $x = \phi(a, d, v)$, so c would get $\phi(c, a, x)$ if matched with a). The right-hand side is the counterpart expression for b. Thus the condition says in effect that a can outbid b in an attempt to match with c instead of d.

If this is true for any value of v then we expect that an equilibrium will never have a matched with d while b is matched with c. But this is all that is meant by PAM: a's partner can never be smaller than b's. In the case of one sided models, taking $c = a$ and $d = b$ gives us segregation: everyone's partner is identical to himself.

Before proceeding, we shall need to establish that equilibria in this environment satisfy an equal treatment property: all agents of the same type receive the same equilibrium payoff. The reason that an argument needs to be made is that this is not a general property of the core in NTU models.[9] But continuity and strictly decreasing frontiers ensure it is satisfied.

Lemma 1 *(Equal Treatment) All agents of the same type receive the same equilibrium payoff.*

Proof. Suppose that there are two agents i and j of type a getting different utilities $v_i > v_j$, and that i's partner k is of type b. Then k gets $\phi(b, a, v_i) < \phi(b, a, v_j)$, where the inequality follows from the fact that ϕ is strictly decreasing in v. By continuity, there exists $\epsilon > 0$ such that

[9] Suppose there are two types, a and b,with the measure of the b's exceeding that of the a's. If an a and a b match, each gets a payoff of exactly 1, while unmatched agents or agents who match with their own type get 0. There is no means to transfer utilty. Then any allocation in which every a is matched to a b, with the remaining b's unmatched, is in the core. But some b's get 1 while others get 0, violating equal treatment.

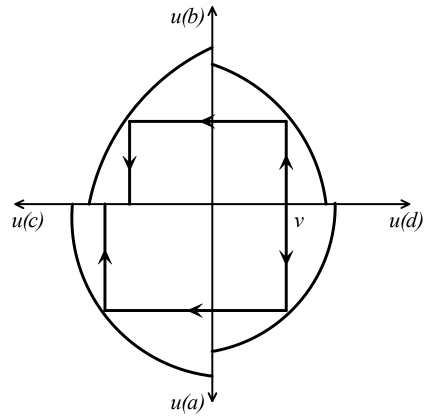

Figure 2: Generalized Increasing Differences

$\phi(b, a, v_j + \epsilon) > \phi(b, a, v_i)$; $\{k, j\}$ can therefore block the equilibrium, a contradiction. ■

This result allows us to refer to a type's payoff without ambiguity.

When satisfied by any v, $a > b$, and $c > d$, condition (5) is called *Generalized Increasing Differences* (GID).[10] The concept is illustrated in Figure 1. The frontiers for the matched pairs $\langle b, d\rangle$,$\langle b, c\rangle$, $\langle a, c\rangle$, and $\langle a, d\rangle$ are plotted in a four-axis diagram. The compositions in (5) are indicated by following the arrows around from a level of utility v for d. Note that the utility c ends up with on the "a side" exceeds that on the b side of the diagram.

Our main result states that GID is sufficient for segregation (PAM in the two-sided case). There is an analogous condition, *Generalized Decreasing Differences* (GDD), for NAM.

Proposition 1 *(1) A sufficient condition for segregation in one-sided models and PAM in two-sided models is generalized increasing differences (GID) on*

[10]The designation Generalized Increasing Differences is motivated as follows. Let A be the type space and G be a (partially) ordered group with operation $*$ and order $\succsim$. We are interested in maps $\psi : A^2 \to G$.

Consider the condition

$$a > b \text{ and } c > d \text{ implies } \psi(c, a) * \psi(d, a)^{-1} \succsim \psi(c, b) * \psi(d, b)^{-1},$$

where $\psi(\cdot, \cdot)^{-1}$ denotes the inverse element under the group operation. When $G = \mathbb{R}$, $\succsim$ = the usual real order, and $*$ = real addition, this is just ID. GID corresponds to the case in which G = monotone functions from $\mathbb{R}$ to itself, $\succsim$ = the pointwise order, and $*$ = functional composition.

$[0, \phi(d,a,0)]$*: whenever* $a > b$*,* $c > d$*, and* $v \in [0, \phi(d,a,0)]$*,* $\phi(c,a,\phi(a,d,v)) \geq \phi(c,b,\phi(b,d,v))$*.*

(2) A sufficient condition for NAM is generalized decreasing differences (GDD) on $[0, \phi(d,b,0)]$*: whenever* $a > b$*,* $c > d$*, and* $v \in [0, \phi(d,b,0)]$*,* $\phi(c,a,\phi(a,d,v)) \leq \phi(c,b,\phi(b,d,v)$*.*

Proof. Here we consider only the one-sided cases; the two-sided cases are similar. For segregation, suppose that instead we have an equilibrium $(\mathfrak{M}, u^*)$ that is not payoff equivalent to segregation: there is a positive measure of heterogeneous matches of the form $\langle a,b \rangle$. Then stability implies a doesn't want to switch to another a, and b doesn't want to switch to b :

$$u_a^* = \phi(a,b,u_b^*) \geq \phi(a,a,u_a^*) = \phi\left(a,a,\phi\left(a,b,u_b^*\right)\right),$$

(here we use equal treatment) and

$$u_b^* \geq \phi(b,b,u_b^*).$$

By payoff nonequivalence, at least one of these inequalities is strict (say it's the first), else matching a with a and b with b with is also an equilibrium. Composing $\phi(a,b,\cdot)$ with the second inequality yields

$$u_a^* \leq \phi(a,b,\phi(b,b,u_b^*)).$$

It then follows that

$$\phi\left(a,a,\phi\left(a,b,u_b^*\right)\right) < \phi(a,b,\phi(b,b,u_b^*))$$

which contradicts the GID condition (taking $c = a$ and $d = b$ there), and we conclude that the economy is segregated.

For one-sided NAM, it suffices to rule out as possible equilibrium matches $(\langle a,b \rangle, \langle c,d \rangle)$ and $(\langle a,c \rangle, \langle b,d \rangle)$ whenever $a > b \geq c > d$, and matches of the form $(\langle a,a \rangle, \langle b,b \rangle)$ for arbitrary $a \neq b$. Suppose to the contrary that $\langle a,b \rangle$ and $\langle c,d \rangle$ is part of a stable match that is not payoff equivalent to NAM. Then

$$\phi(a,b,u_b^*) \geq \phi(a,d,u_d^*)$$

(a weakly prefers b to d) and

$$u_b^* \geq \phi(b,c,u_c^*) = \phi(b,c,\phi(c,d,u_d^*)) \tag{6}$$

(b weakly prefers a to c). At least one of these is strict; assume it's the first. Apply $\phi(b,a,\cdot)$ to the strict form of the first inequality to get

$$u_b^* < \phi(b,a,\phi(a,d,u_d^*)). \tag{7}$$

Combining (6) and (7) yields

$$\phi(b,a,\phi(a,d,u_d^*)) > \phi(b,c,\phi(c,d,u_d^*)),$$

contradicting GDD.

If instead $\langle a,c\rangle$ and $\langle b,d\rangle$ are stable, we have

$$\phi(a,c,u_c^*) > \phi(a,d,u_d^*) \Longrightarrow u_c^* < \phi(c,a,\phi(a,d,u_d^*))$$

and

$$u_c^* \geq \phi(c,b,\phi(b,d,u_d^*)),$$

which again contradicts GDD.

Finally, if $\langle a,a\rangle$ and $\langle b,b\rangle$ are stable, and without loss of generality $a > b$, then the equilibrium payoffs satisfy $u_a^* = \phi(a,a,u_a^*)$ and $u_b^* = \phi(b,b,u_b^*)$ by equal treatment, and $u_a^* > \phi(a,b,u_b^*)$, by stability and payoff nonequivalence. Apply $\phi(a,a,\cdot)$ to this inequality to get

$$\phi(a,a,u_a^*) < \phi(a,a,\phi(a,b,u_b^*)).$$

GDD implies

$$\phi(a,a,\phi(a,b,u_b^*)) \leq \phi(a,b,\phi\left(b,b,u_b^*\right)).$$

Thus

$$u_a^* > \phi(a,b,u_b^*) = \phi(a,b,\phi(b,b,u_b^*)) \geq \phi(a,a,\phi(a,b,u_b^*)) > u_a^*,$$

a contradiction. ■

We now apply this result our model of risk sharing within households.

Example 5 *We claim that the GDD is satisfied in the risk sharing example. Recall from (2) that* $\phi(a,b,v) = \ln(1-e^{v-\Sigma_{ab}}) + \Sigma_{ab}$*, where* Σ_{ab} *denotes* $\Sigma_i \pi_i \ln(w_i + a + b + 2)$*. Now let* $a > b$ *and* $c > d$*. Then*

$$\begin{aligned}\phi(c,a,\phi(a,d,v) &= \ln(1-e^{\ln(1-e^{v-\Sigma_{ad}})+\Sigma_{ad}-\Sigma_{ac}}) + \Sigma_{ac} \\ &= \ln(1-e^{\Sigma_{ad}-\Sigma_{ac}} + e^{v-\Sigma_{ac}}) + \Sigma_{ac}\end{aligned}$$

and

$$\phi(c,b,\phi(b,d,v) = \ln(1-e^{\Sigma_{bd}-\Sigma_{bc}} + e^{v-\Sigma_{bc}}) + \Sigma_{bc}.$$

Now,

$$\phi(c, a, \phi(a, d, v)) \ < \ \phi(c, b, \phi(b, d, v))$$

if and only if

$$(1 - e^{\Sigma_{ad} - \Sigma_{ac}} + e^{v - \Sigma_{ac}})e^{\Sigma_{ac}} < (1 - e^{\Sigma_{bd} - \Sigma_{bc}} + e^{v - \Sigma_{bc}})e^{\Sigma_{bc}},$$

that is if $e^{\Sigma_{ac}} - e^{\Sigma_{ad}} < e^{\Sigma_{bc}} - e^{\Sigma_{bd}}$*. This is just the requirement that the function* $e^{\Sigma_{ab}}$ *satisfies decreasing differences, which it clearly does, since* $\frac{\partial^2}{\partial a \partial b} e^{\Sigma_{ab}} = -e^{\Sigma_{ab}} Var(\frac{1}{w+a+b+2}) < 0$.
Thus GDD is indeed satisfied, and we conclude that in the risk-sharing economy with logarithmic utility, agents will always match negatively in wealth. *This is intuitive: a risk-neutral agent is willing to offer a better deal for insurance than is a risk averse one, so those demanding the most insurance (the most risk averse, i.e., the poor) will share risk with the least risk averse (the rich), while the moderately risk averse share with each other.*

5 Computational Aids

A number of useful computational techniques follow from the sufficiency of the GID and GDD. We first present a set of differential conditions. In addition to being easy to apply, they help sharpen the intuition about the trade-offs at work in NTU matching problems.

Next we note that GID and GDD are preserved under ordinal transformations of types' preferences. This implies that the analyst is free to choose whichever representation of preferences is most convenient, and leads to a weakening of the differential conditions. In case the NTU model admits a TU representation, GID and GDD reduce to ID and DD of the joint payoff function.

Finally, we develop the lattice-theoretic versions of our conditions and conclude he section with a remark on models with type-dependent autarchy payoffs.

5.1 Differential Conditions

Often it is easier to check whether a condition holds locally than globally, particularly if a closed-form expression for the frontier is not available. We now provide a set of local conditions which suffice for monotone matching. In addition to being computationally convenient, these conditions illuminate the "complementarity in transferability" property alluded to above. In this subsection we suppose that $\phi(x, y, v)$ is twice differentiable (except of course at $v = \phi(y, x, 0)$).

Proposition 2 *(i) A sufficient condition for segregation (or PAM) is that for all $x, y \in A \times A$ and $v \in [0, \phi(y, x, 0))$,*

$$\phi_{12}(x, y, v) \geq 0, \phi_{13}(x, y, v) \geq 0 \text{ and } \phi_1(x, y, v) \geq 0. \tag{8}$$

(ii) A sufficient condition for NAM is that for all $x, y \in A \times A$ and $v \in [0, \phi(y, x, 0))$,

$$\phi_{12}(x, y, v) \leq 0, \phi_{13}(x, y, v) \leq 0 \text{ and } \phi_1(x, y, v) \geq 0. \tag{9}$$

Proof. We show that the local conditions imply the generalized difference conditions. Fix v, $a > b$ and $c > d$, and consider the case (i) for segregation/PAM (the other case is similar). Then $\phi_{12} \geq 0$ implies that for any $t \in [d, c]$

$$\phi_1(t, a, \phi(b, d, v)) \geq \phi_1(t, b, \phi(b, d, v));$$

$\phi_1 \geq 0$ implies $\phi(a, d, v) \geq \phi(b, d, v)$, and $\phi_{13} \geq 0$ in turn yields

$$\phi_1(t, a, \phi(a, d, v)) \geq \phi_1(t, a, \phi(b, d, v)),$$

so that

$$\phi_1(t, a, \phi(a, d, v)) \geq \phi_1(t, b, \phi(b, d, v)).$$

Integrating both sides of this inequality over t from d to c then gives

$$\phi(c, a, \phi(a, d, v)) - \phi(d, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v)) - \phi(d, b, \phi(b, d, v));$$

Noting that $\phi(d, a, \phi(a, d, v)) = \phi(d, b, \phi(b, d, v)) = v$ gives us GID. ■

Obviously, with TU, $\phi_{13} = 0$, so this reduces to the standard condition in that case. The extra term reflects the fact that changing the type results in a change in the slope of the frontier. For segregation/PAM, the idea is that higher types can transfer utility to their partners more easily (ϕ_3 is less negative, hence flatter).

The conditions in Proposition 2 illustrate the separate roles of both the usual "productivity" complementarity and the "transferability" complementarity we have mentioned. In terms of the bidding story we mentioned in

Section 4.1, if two different types are competing for a higher partner, both will have to offer her more than they would a lower partner ($\phi_1 > 0$); if the higher type's frontier is flatter than the lower's frontier ($\phi_{13} \geq 0$), it will cost the higher type less to do this than it will the lower one; meanwhile if the high type is also more productive on the margin ($\phi_{12} > 0$) then he is sure to win, in effect being both more productive and having lower costs.

To be sure, it is not necessary for the two effects to be operative in the same direction: for segregation/PAM one only needs the net effect to be positive. Indeed, the conditions in Proposition 2 imply that the potential utility gains from an increase in one's attribute are monotonic in the partner's attribute: if (8) holds,

$$\frac{d}{da}\phi_1(t,a,\phi(a,t,v)) = \phi_{12}(d,b,\phi(b,d,v)) + \phi_{13}(d,b,\phi(b,d,v)) \cdot \phi_1(b,d,v) \geq 0, \tag{10}$$

and if (9) does, then

$$\phi_{12}(d,b,\phi(b,d,v)) + \phi_{13}(d,b,\phi(b,d,v)) \cdot \phi_1(b,d,v) \leq 0; \tag{11}$$

but the reverse implications are not true.[11]

Closely related conditions are sufficient for monotone matching, if perhaps harder to verify than (8) and (9). Like (10) and (11), they involve compositions of ϕ and its partial derivatives; we simply mention them without further comment; the proof is similar to that of Proposition 2.

Proposition 3 *If ϕ is smooth, a sufficient condition for segregation/PAM is that for all types x, y, z with $z \leq x$, and utilities v,*

$$\phi_{12}(x,y,\phi(y,z,v)) + \phi_{13}(x,y,\phi(y,z,v)) \cdot \phi_1(y,z,v) \geq 0.$$

A sufficient condition for NAM is that for all types x, y, z with $z \leq x$, and utilities v,

$$\phi_{12}(x,y,\phi(y,z,v)) + \phi_{13}(x,y,\phi(y,z,v)) \cdot \phi_1(y,z,v) \leq 0.$$

[11] In fact, (10) and (11) are implied by the generalized difference conditions. To see this, take $a > b$ and $c > d$ and note that GID is equivalent to $\phi(c,a,\phi(a,d,v)) - \phi(d,a,\phi(a,d,v)) \geq \phi(c,b,\phi(b,d,v)) - \phi(d,b,\phi(b,d,v))$. Dividing by $c-d$ and taking limits as $c \to d$ yields $\phi_1(d,a,\phi(a,d,v)) \geq \phi_1(d,b,\phi(b,d,v))$. Dividing by $a-b$ and letting $a \to b$ yields

$$\phi_{12}(d,b,\phi(b,d,v)) + \phi_{13}(d,b,\phi(b,d,v)) \cdot \phi_1(b,d,v) \geq 0,$$

as claimed. The GDD case is similar.

The condition $\phi_1 \geq 0$ in Propositions 2 and 3 is less restrictive than might first appear: in a model in which instead $0 \geq \phi_1$ everywhere, one can redefine the type space with the "reverse" order; then the cross partial ϕ_{12} retains its sign, while ϕ_{13} and ϕ_1 reverse sign and Proposition 2 can be applied (in other words, if $\phi_1 \leq 0$ everywhere, then monotone matching occurs when ϕ_{12} and ϕ_{13} are opposite-signed).

Finally, we show in the next subsection that the differential conditions can be weakened further by considering increasing transformations of preferences.

Example 6 *Earlier we conjectured that the most risk averse agents ought to match with the most well monitored tasks, since the latter are optimally contracted as fixed wages. This intuition is incomplete, and indeed misleading, as the following application of Proposition 2 indicates.*

Recall from (3) and (4) that

$$\phi(p, a, v) = \pi + a - e^{v+1}[pe^{\frac{1-p}{2p-1}} + (1-p)e^{-\frac{p}{2p-1}}]$$

and

$$\phi(a, p, v) = \frac{1-p}{2p-1} + \ln\left(\frac{\pi + a - v}{pe^{\frac{1}{2p-1}} + 1 - p}\right).$$

Thus, when own type is a principal,

$$\phi_1(p, a, v) = \phi_{13}(p, a, v) = \left(e^{\frac{1-p}{2p-1}}\left(\frac{p}{(2p-1)^2} - 1\right) - e^{-\frac{p}{2p-1}}\left(\frac{1-p}{(2p-1)^2} - 1\right)\right)e^{v+1} > 0,$$

and when own type is an agent, $\phi_1(a, p, v) = \frac{1}{\pi+a-v} > 0$ *and* $\phi_{13}(a, p, v) = \left(\frac{1}{\pi+a-v}\right)^2 > 0$*. Moreover,* $\phi_{12} = 0$ *in either case.*

Thus the agents with lower risk aversion (higher wealth) are matched to principals with higher quality signals*, i.e. more observable tasks. This result may appear surprising, since empirically we tend to associate* less *observable tasks to wealthier workers (in particular one would expect the poor to take fixed wages while the rich bear risk).*

The explanation for the result is that in the standard version of the principal-agent model with utility additively separable in income and effort, incentive compatibility for a given effort level entails that the amount of risk borne by the agent increases with wealth (in this case, the variance of the agent's income is $p(1-p)e^{\frac{2}{2p-1}}\left(\frac{\pi+a-v}{pe^{\frac{1}{2p-1}}+1-p}\right)^2$*). This effect arises from the diminishing marginal utility of income. Though wealthier agents tolerate risk* *better than the poor, they must accept more risk on a given task; with logarithmic utility (and indeed for many other utilities – see Newman, 1999), the latter effect dominates, and the wealthy therefore prefer the safer tasks. Put another way, better monitoring allows for a reduction in risk borne by the agent; given the increasing risk effect of incentive compatibility, the benefit of the risk reduction is greater for the rich than for the poor, and this generates a complementarity between monitoring and wealth.*

The result offers a possible explanation for the finding in Ackerberg-Botticini (2002) that in medieval Tuscany, wealthy peasants were more likely than poor peasants to tend safe *crops (cereals) rather than risky ones (vines).*

This example is instructive because the entire effect comes from the non-transferability of the problem. There is no direct "productive" interaction between principal type and agent type ($\phi_{12} = 0$); only the complementarity between type and transferability plays a role in determining the match.

Finally, as is apparent from their derivation, the local conditions are stronger than generalized difference conditions, even restricting to smooth frontier functions. This is of practical as well as logical interest: as we saw, Example 3 satisfies GDD, from which we concluded there is negative matching in wealth. But in spite being smooth, $\phi(a, b, v) = \ln(1 - e^{v - \Sigma_i \pi_i \ln(w_i + a + b)}) + \Sigma_i \pi_i \ln(w_i + a + b) \equiv \ln(1 - e^{v - \Sigma_{ab}}) + \Sigma_{ab}$ doesn't satisfy the local condition:

$$\phi_1 = \frac{1}{1 - e^{v - \Sigma_{ab}}} \frac{\partial \Sigma_{ab}}{\partial a} > 0,$$

$$\phi_{12} = \frac{1}{(1 - e^{v - \Sigma_{ab}})^2} \left((1 - e^{v - \Sigma_{ab}}) \frac{\partial^2 \Sigma_{ab}}{\partial a \partial b} - e^{v - \Sigma_{ab}} \left(\frac{\partial \Sigma_{ab}}{\partial a} \right)^2 \right) < 0,$$

yet

$$\phi_{13} = \frac{e^{v - \Sigma_{ab}}}{(1 - e^{v - \Sigma_{ab}})^2} \frac{\partial \Sigma_{ab}}{\partial a} > 0.$$

Thus, for some models, the generalized difference conditions may apply while the local conditions do not.[12] However, in such cases, it may be possible to find an alternate representation of agents' preferences in which the frontiers do satisfy the local conditions.

[12] Of course, (11) is satisfied for this example, as it must be since it is a consequence of GDD:

5.2 Order Preserving Transformations

The core of an economy is independent of the cardinal representation of preferences; in particular the matching pattern must not depend on how one represents the preferences of the agents. However, so far we have only established sufficiency of the generalized difference conditions for monotone matching, and so it is legitimate to ask whether they hold after monotone transformations of types' utilities.

Suppose that

$$\phi(a,b,v) = \max_{x,x'} U(x,a) \text{ s.t. } U(x',b) \geq v,$$

where x and x' are choice variables taken to be in some feasible set. Let h be an increasing transformation applied to one type's utility, say t: we replace $U(x,t)$ by $h(U(x,t))$, $\phi(t,b,v)$ by $h(\phi(t,b,v))$ and $\phi(b,t,v)$ by $\phi(b,t,h^{-1}(v))$. We have the following result:

Proposition 4 *Suppose GID (GDD) holds for ϕ. Then GID (GDD) holds for any other frontier function generated from ϕ by monotone transformations of types' utilities.*

Proof. We consider the case for GID; the proof for GDD is virtually identical. Note that it is enough to show that if GID holds for ϕ, then it holds for the frontier function derived by transforming a single type's utility; the proposition is verified by repeating the argument for all types.

To show that GID also holds for the new frontier function, suppose first that $t = c$ in the expression

$$a > b, c > d, v \in [0, \phi(d,b,0)] \Longrightarrow \phi(c,a,\phi(a,d,v)) \geq \phi(c,b,\phi(b,d,v)).$$

$$\begin{aligned}
& \phi_{12}(a,b,\phi(b,a,v)) + \phi_{13}(a,b,\phi(b,a,v)) \cdot \phi_1(b,a,v) \\
= \; & \frac{1}{(1-e^{\ln(1-e^{v-\Sigma_{ab}})+\Sigma_{ab}-\Sigma_{ab}})^2} \left((1-e^{\ln(1-e^{v-\Sigma_{ab}})+\Sigma_{ab}-\Sigma_{ab}}) \frac{\partial^2 \Sigma_{ab}}{\partial a \partial b} - e^{\ln(1-e^{v-\Sigma_{ab}})+\Sigma_{ab}-\Sigma_{ab}} \left(\frac{\partial \Sigma_{ab}}{\partial a} \right)^2 \right) \\
& + \frac{e^{\ln(1-e^{v-\Sigma_{ab}})+\Sigma_{ab}-\Sigma_{ab}}}{(1-e^{\ln(1-e^{v-\Sigma_{ab}})+\Sigma_{ab}-\Sigma_{ab}})^2} \frac{\partial \Sigma_{ab}}{\partial a} \cdot \frac{1}{1-e^{v-\Sigma_{ab}}} \frac{\partial \Sigma_{ab}}{\partial a} \\
= \; & \left(\frac{1}{e^{v-\Sigma_{ab}}} \right)^2 \left(e^{v-\Sigma_{ab}} \frac{\partial^2 \Sigma_{ab}}{\partial a \partial b} - (1-e^{v-\Sigma_{ab}}) \left(\frac{\partial \Sigma_{ab}}{\partial a} \right)^2 \right) + \left(\frac{1}{e^{v-\Sigma_{ab}}} \right)^2 \left(\frac{\partial \Sigma_{ab}}{\partial a} \right)^2 \\
= \; & \frac{1}{e^{v-\Sigma_{ab}}} \left(\frac{\partial^2 \Sigma_{ab}}{\partial a \partial b} + \left(\frac{\partial \Sigma_{ab}}{\partial a} \right)^2 \right) = -e^{\Sigma_{ab}-v} Var(\frac{1}{w+a+b+2}) < 0.
\end{aligned}$$

Then we simply need to apply h to both sides of the implied inequality, which obviously preserves its truth value. If $t = a$, then only the left hand side is affected: we get $\phi(c, a, h^{-1}(h(\phi(a, d, v)))) = \phi(c, a, \phi(a, d, v))$, so again the GID holds under the new representation of preferences. And if $t = d$, then we need

$$z \in [h(0), h(\phi(d, b, 0)] \Longrightarrow \phi(c, a, \phi(a, d, h^{-1}(z))) \geq \phi(c, b, \phi(b, d, h^{-1}(z))),$$

which follows from the fact that $h^{-1}(z)$ always lies in $[0, \phi(d, b, 0)]$. ■

We shall call a frontier function $\hat{\phi}$ generated from ϕ by subjecting all types' utilities to increasing transformations a *representation* of ϕ. A suitably chosen representation of ϕ may be easier to work with than ϕ itself:

Example 7 *In the risk sharing example the frontier was computed to be* $\phi(a, b, v) = \ln(1 - e^{v - \Sigma_{ab}}) + \Sigma_{ab}$, *where* $\Sigma_{ab} \equiv \sum \pi_i \ln(a + b + w_i + 2)$. *If we transform the utility by exponentiation (doing so for all types), we get* $\hat{\phi}(a, b, v) = e^{\phi(a,b,\ln v)} = e^{\ln(1 - e^{\ln v - \Sigma_{ab}}) + \Sigma_{ab}} = e^{\Sigma_{ab}} - v$. *This function appears to be more manageable than* ϕ; *indeed* $\hat{\phi}$ *is a* transferable utility representation *of* ϕ *(more on this in the next subsection). GDD for* $\hat{\phi}$ *is satisfied:* $\hat{\phi}(c, a, \hat{\phi}(a, d, v)) < \hat{\phi}(c, b, \hat{\phi}(b, d, v)) \Longleftrightarrow e^{\Sigma_{ca}} - (e^{\Sigma_{ad}} - v) < e^{\Sigma_{cb}} - (e^{\Sigma_{bd}} - v) \Longleftrightarrow e^{\Sigma_{ab}}$ *satisfies DD, which we verified earlier.*

Though the generalized difference conditions are preserved for all representations of ϕ, not so the differential conditions in Proposition 2. Recall that in the risk sharing example, those conditions do not hold for ϕ. But they do for the above transformed version of $\hat{\phi}$: indeed, $\hat{\phi}_1 > 0$, $\hat{\phi}_{12} < 0$, and $\hat{\phi}_{13} = 0$. This suggests the following strengthening of Proposition 2, whose proof is an immediate consequence of the fact that the differential conditions imply GID, which in turn implies GID of any representation of ϕ.

Corollary 1 *(1) A sufficient condition for segregation (or PAM) is that there exists a representation* $\hat{\phi}$ *of* ϕ *such that for all* $x, y \in A \times A$ *and* $v \in [0, \phi(y, x, 0))$,

$$\hat{\phi}_{12}(x, y, v) \geq 0, \hat{\phi}_{13}(x, y, v) \geq 0 \text{ and } \hat{\phi}_1(x, y, v) \geq 0.$$

(2) A sufficient condition for NAM is that there exists a representation $\hat{\phi}$ *of* ϕ *such that for all* $x, y \in A \times A$ *and* $v \in [0, \phi(y, x, 0))$,

$$\hat{\phi}_{12}(x, y, v) \leq 0, \hat{\phi}_{13}(x, y, v) \leq 0 \text{ and } \hat{\phi}_1(x, y, v) \geq 0.$$

5.3 TU Representability

We noted that by transforming the payoffs of the agents in the risk sharing example, we could express the frontiers in a transferable utility form. This cannot be done with all NTU models, of course (see Legros-Newman, 2003 for more on this topic), but there are some well-known-examples. For instance, the Principal-Agent model with exponential utility (Holmstrom-Milgrom, 1987) can be given a TU representation by looking at players' certainty equivalent incomes rather than their expected utility levels. Another instance is the principal-agent example in this paper.

Start with a model $\phi(a, b, v)$ and say that it is *TU-representable* if there is a set of increasing transformations $F(t, \cdot)$, indexed by type t, and a function of types $\psi(t, t')$ such that

$$\forall a, b, v,\ F(a, \phi(a, b, v)) = \psi(a, b) - F(b, v).$$

Then $F(a, \phi(a, b, v))$ is a TU model, since the transformed payoffs to (a, b) sum to $\psi(a, b)$, independently of the distribution of transformed utility between a and b. It follows from the definition that ψ is symmetric.[13] The main observation of this subsection is the following

Proposition 5 *Suppose that ϕ has a TU representation (F, ψ). Then ϕ satisfies GID (GDD) if and only if ψ satisfies ID (DD).*

Proof. Take $a > b$, $c > d$, and v and assume GID holds. Then

$$\phi(c, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v))$$

$$\begin{aligned}
&\iff F(c, \phi(c, a, \phi(a, d, v))) \geq F(c, \phi(c, b, \phi(b, d, v))) \\
&\iff \psi(c, a) - F(a, \phi(a, d, v)) \geq \psi(c, b) - F(b, \phi(b, d, v)) \\
&\iff \psi(c, a) - \psi(a, d) + F(d, v) \geq \psi(c, b) - \psi(b, d) + F(d, v) \\
&\iff \psi(c, a) - \psi(d, a) \geq \psi(c, b) - \psi(d, a),
\end{aligned}$$

i.e. ψ satisfies increasing differences. The proof for GDD simply reverses all the weak inequalities. ■

Example 8 *For the principal agent example, put* $F(p, v) = \frac{1}{pe^{\frac{1-p}{2p-1}} + (1-p)e^{\frac{-p}{2p-1}}} v$, *and* $F(a, v) = e^{v+1}$. *The sum* $F(p, \phi(p, a, v)) + F(a, v)$ *is then* $\psi(p, a) = \frac{\pi + a}{pe^{\frac{1-p}{2p-1}} + (1-p)e^{\frac{-p}{2p-1}}}$ *, which satisfies ID: matching is always positive assortative.*

[13] To see this, note that $F(b, \phi(b, a, v)) = \psi(b, a) - F(a, v)$, but $F(a, v) = F(a, \phi(a, b, \phi(b, a, v))) = \psi(a, b) - F(b, \phi(b, a, v))$, so that $F(b, \phi(b, a, v)) = \psi(a, b) - F(a, v)$; hence $\psi(a, b) = \psi(b, a)$.

5.4 Lattice Theoretic Conditions

Proposition 2 can be weakened by considering (possibly) nondifferentiable functions that are supermodular in pairs of variables.

Proposition 6 *(1) A sufficient condition for segregation (PAM in two sided models) is that ϕ is supermodular in types, increasing in own type, and supermodular in own type and payoff.*
(2) A sufficient condition for NAM is that ϕ is submodular in types, increasing in own type and submodular in own type and payoff.

Proof. Consider case (1); the other case is similar. Take v, $a > b$ and $c > d$. Supermodularity in own type and partner's utility, along with increasing in own type implies $\phi(c, a, \phi(a, d, v)) + \phi(d, a, \phi(b, d, v)) \geq \phi(c, a, \phi(b, d, v)) + \phi(d, a, \phi(a, d, v))$, or $\phi(c, a, \phi(a, d, v)) - \phi(d, a, \phi(a, d, v)) \geq \phi(c, a, \phi(b, d, v)) - \phi(d, a, \phi(b, d, v))$. But the right hand side of the latter inequality weakly exceeds $\phi(c, b, \phi(b, d, v)) - \phi(d, b, \phi(b, d, v))$ by supermodularity in types. Thus $\phi(c, a, \phi(a, d, v)) - \phi(d, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v)) - \phi(d, b, \phi(b, d, v))$, and since $\phi(d, a, \phi(a, d, v)) = \phi(d, b, \phi(b, d, v)) = v$, $\phi(c, a, \phi(a, d, v)) \geq \phi(c, b, \phi(b, d, v))$, which is GID. ■

It is evident from this proposition that a stronger sufficient condition for segregation (or PAM) is that ϕ itself is a supermodular function that is increasing in own type, since this implies the condition in Proposition 6.[14]

The principal interest of this observation is that it enables us to offer sufficient conditions for monotone matching expressed in terms of the fundamentals of the model, rather than in terms of the frontiers (such results leading to our local conditions would be much harder to come by).

The frontier can be expressed fairly generally as

$$\begin{aligned} \phi(a, b, v) &= \max_{x,x'} U(x, a) \\ \text{s.t. } U(x', b) &\geq v \\ (x, x') &\in \Phi(a, b). \end{aligned}$$

Here $\Phi(a, b) \subset X$, a (sub)lattice of some $\mathbb{R}^n$, is the set of choices available to types (a, b). A sufficient condition for ϕ to be increasing in own type

[14] More directly, given v, $a > b$ and $c > d$, put $x = (d, a, \phi(a, d, v))$ and $y = (c, b, \phi(b, d, v))$ in the defining inequality $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \geq \phi(\mathbf{x}) + \phi(\mathbf{y})$. Then since $\phi(a, d, v) \geq \phi(b, d, v)$, $\mathbf{x} \vee \mathbf{y} = (c, a, \phi(a, d, v))$, $\mathbf{x} \wedge \mathbf{y} = (d, b, \phi(b, d, v))$,and we have

$$\phi(c, a, \phi(a, d, v)) + \phi(d, b, \phi(b, d, v)) \geq \phi(d, a, \phi(a, d, v)) + \phi(c, b, \phi(b, d, v)),$$

which is just GID since $\phi(d, b, \phi(b, d, v)) = \phi(d, a, \phi(a, d, v) = v$.

is that U is increasing in type and Φ is continuous and increasing (in the set inclusion order) in own type. A sufficient condition for ϕ to be strictly decreasing in v is that U is strictly monotone in x.

We also need the set

$$S = \{(a, b, v, x, x')|a \in A, b \in A, v \in \mathbb{R}, (x, x') \in \Phi(a, b)\}$$

to form a sublattice. Then an application of Theorem 2.7.2 of Topkis (1998) yields

Corollary 2 *If payoffs functions are supermodular (submodular), strictly increasing in choices, and increasing in type; choice sets are continuous and increasing in own type; and the set of types, payoffs and feasible choices forms a sublattice, then the economy is segregated in the one-sided case and positively matched in the two-sided case (negatively matched).*

Topkis's theorem tells us that under the stated hypotheses, ϕ will be supermodular (submodular); since it is also increasing in own type by the hypotheses on F and U, the result follows.

As a practical matter, the usefulness of this corollary hinges on the ease of verifying that the sets S and F have the required properties. In many cases it may be more straightforward to compute the frontiers and apply Propositions 1, 2, or 6. Note, for example, that since the frontier function in the risk-sharing example is not submodular despite the fact that the objective function is, the choice-parameter set S is not a sublattice.[15]

5.5 Type-Dependent Autarchy Payoffs

Suppose that autarchy generates a payoff $\underline{u}(a)$ to type a; if A is compact and $\underline{u}(\cdot)$ continuous, without loss of generality, we can assume $\underline{u}(a) \geq 0$.

[15] One might also wonder about the relationship between GID and quasi-supermodularity (QSM) (Milgrom-Shannon, 1994; Topkis, 1998). In terms of the notation in footnote 10, the map $\psi : A^2 \to G$ is quasi-supermodular if $\psi(x) \succsim \psi(x \wedge y) \Longrightarrow \psi(x \vee y) \succsim \psi(y)$, with the same implication holding for the strict order; putting $x = (b, c)$ and $y = (a, d)$ this can be stated as $a > b$, $c > d$, and $\psi_{bc} \succsim \psi_{bd}$ implies $\psi_{ac} \succsim \psi_{ad}$. Equivalently, we must have $\psi_{cb} \circ \psi_{bd} \succsim Id$ implies $\psi_{ca} \circ \psi_{ad} \succsim Id$, where Id is the identity map. GID of ψ clearly implies QSM of ψ, but not vice versa: suppose that for some types a, b, c, d and payoff v we have $\phi(c, a, \phi(a, d, v)) < \phi(c, b, \phi(b, d, v)) < v$; thus ϕ (and ψ) violates GID, but ψ does not violate QSM. Moreover, QSM is not sufficient for PAM: if QSM is satisfied and GID is not, as in this case, there is a type distribution for which the equilibrium is NAM rather than PAM (the logic parallels that in Section 5.6). A similar set of relationships can be demonstrated if we ask for QSM of ϕ rather than ψ.

Then all the propositions go through as before, since if the generalized difference or differential conditions hold for nonnegative payoffs, they hold on the restricted domain of individually rational ones. Equilibrium will now typically entail that some types remain unmatched (even apart from excess supply issues), but *among those matched*, the pattern will be monotone if the appropriate difference condition holds.

5.6 Necessity

A natural issue to consider at this point is the strength of the sufficient conditions we have given for monotone matching: is GID necessary for PAM? Here we can give an affirmative answer for the two-sided case:

Proposition 7 *In a two-sided model, if the equilibrium outcome is PAM (NAM) for all distributions of types, then the frontier function ϕ satisfies GID (GDD).*

Proof. Consider PAM, as the case for the necessity of GDD for NAM is similar. Suppose there exist $a > b$ on one side and $c > d$ on the other, and a payoff level v such that $\phi(c, a, \phi(a, d, v)) < \phi(c, b, \phi(b, d, v))$. Then we can find a distribution of types such that there is an equilibrium that is not payoff equivalent to PAM. To see this, put an equal measure at each of the four types a, b, c, d. Then there is $\epsilon > 0$ such that $\langle a, d \rangle$ with payoffs $(\phi(a, d, v), v)$ and $\langle b, c \rangle$ with payoffs $(\phi(b, d, v) + \epsilon, \phi(c, b, \phi(b, d, v) + \epsilon))$ is an equilibrium. To verify stability, note that by continuity of ϕ in v, for ϵ small enough, $\phi(c, a, \phi(a, d, v)) < \phi(c, b, \phi(b, d, v) + \epsilon)$. Thus c would be strictly worse off switching to a as long as a receives at least his equilibrium payoff; similarly d would do strictly worse to switch to b. ■

The one-sided case is a bit more involved. It is known that in the one-sided *TU* model that ID is not necessary for segregation, and that this condition can be weakened to nonpositivity of a function derived from the joint payoff called the *surplus*. One-sided PAM (outside of the definition of which we have not considered here) and NAM are equivalent to something called *weak increasing differences* and *weak decreasing differences* of this derived function (Legros-Newman, 2002a).

When utility is nontransferable, a similar construction can be performed in which a surplus function is derived from the frontier ϕ; suitably weakened versions of the generalized difference conditions defined for the surplus function are then necessary as well as sufficient for monotone matching. The interested reader is referred to Legros-Newman (2002c).

6 Conclusion

6.1 Summary

Many economic situations involving nontransferable utility are naturally modeled as matching or assignment games. For these to have much use, it is necessary to characterize equilibrium matching patterns. We have presented some general sufficient conditions for monotone matching in these models. These have an intuitive basis and appear to be reasonably straightforward to apply. Specifically, if one wants to ensure PAM, it does not suffice only to have complementarity in types of productivity; one must ensure as well that there is enough complementarity of type and transferability.

To summarize, if one wants to check that equilibrium matching pattern of an NTU model with continuous, decreasing Pareto frontiers is monotone:

- Check that the model satisfies GID for segregation/PAM, and GDD for NAM.
- If this proves unworkable, try the differential conditions (or their lattice theoretic counterparts).
- Take advantage of the ordinal nature of GID; perhaps a monotone transformation of types' payoffs is tractable enough that GID or the differential conditions can be verified
- In particular, the model may have a TU representation, in which case one need only check for ID or DD of the joint payoff.

6.2 Discussion

This paper has focused on the study of properties of the economic environment that lead to monotone matching. Implicitly motivating this analysis is the question of how changes in the environment influence changes in matching. Space, not to mention the present state of knowledge, is too limited to offer a complete answer to this question here, but the comparison of TU with NTU is no doubt an important first step. Here we simply point out that economy-wide changes to transferability may help to explain phenomena that could be characterized as mass re-assignments of partners.

For instance, mergers and divestitures involve reassignments of say, upstream and downstream divisions of firms. Transferability between divisions depends on the efficiency of credit markets, and that in turn depends on interest rates– higher ones lead to an increase in agency costs, i.e. reductions in transferability, with the magnitude of the effect dependent on characteristics of individual firms such as liquidity position or productivity. A shock to the interest rate then may lead to widespread reassignment of partnerships between upstream and downstream divisions, i.e., a "wave" of corporate reorganization (Legros-Newman, 1999).

Or consider the effects of a policy like Title IX, which requires US schools and universities receiving federal funding to spend equally on men's and women's activities (athletic programs having garnered the most public attention), or suffer penalties in the form of lost funding. If one models a college as partnership between a male and female student-athlete, identifying their types with the revenue-generating capacities of their respective sports, the policy acts to transform a TU model into an NTU one, rather like Example 1. Imposing Title IX would lead to a reshuffling of the types males and females who match; the male wrestler (low revenue), formerly matched to the female point guard (high revenue), will now match with, say, a female rower, while the point guard now plays at a football school. There is evidence that this sort of re-assignment has taken place: the oft-noted terminations and contractions of some sports at some colleges are ameliorated by start-ups and expansions at others.

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